EXAM ALGEBRAIC STRUCTURES,

July 12th, 2019, 9.00am–12.00am, NB 5113.0201.

Please provide complete arguments for each of your answers. The exam consists of 3 questions each subdivided into 4 parts. You can score up to 3 points for each part, and you obtain 4 points for free.

In this way you will score in total between 4 and 40 points.

(1) In this exercise we denote, for any $n \in \mathbb{Z}$, the ring $\mathbb{Z}[t]/(t^2 - n)$ by R_n. Elements of R_n we write as $f(t) \bmod (t^2 - n)$, given any $f(t) \in \mathbb{Z}[t]$.
  - (a) Show $(t+1) \bmod (t^2 - n) \in R_n^\times \iff n \in \{0, 2\}$.
  - (b) Show that R_n is an integral domain if and only if n is *not* a square in $\mathbb{Z}$.
  - (c) Show that R_n contains a nilpotent element different from the zero element if and only if $n = 0$.
  - (d) Show that $R_2 \cong \mathbb{Z}[\sqrt{2}]$, and prove that this ring is a Euclidean domain, with function $g\colon \mathbb{Z}[\sqrt{2}] \to \mathbb{Z}_{\geq 0}$ given by $g(n + m\sqrt{2}) = |n^2 - 2m^2|$.

(2) Let $\alpha \in \mathbb{R}$ be a root of the equation $x^2 + 3x = 1$ and consider the ring $\mathbb{Z}[\alpha] := \{a + b\alpha \mid a, b \in \mathbb{Z}\} \subset \mathbb{R}$.
  - (a) Show that $N\colon \mathbb{Z}[\alpha] \to \mathbb{Z}$ given by $N(a + b\alpha) = a^2 - 3ab - b^2$ is multiplicative; i.e., $N(xy) = N(x)N(y)$ for $x, y \in \mathbb{Z}[\alpha]$.
  - (b) Show that $\varphi\colon \mathbb{Z}[\alpha] \to \mathbb{Z}/3\mathbb{Z}$ is a ring homomorphism with kernel the ideal generated by 3 and $\alpha - 1$.
  - (c) Show that the ideal given in (b) is a principal ideal and moreover that it is maximal.
  - (d) Is $\alpha - 2 \in \mathbb{Z}[\alpha]$ irreducible?

(3) This final exercise considers a finite field.
  - (a) List all irreducible elements in $\mathbb{F}_2[t]$ of degree ≤ 2 (explain why your list is complete!).
  - (b) Show that $t^5 + t + 1 \in \mathbb{F}_2[t]$ is reducible and $t^5 + t^2 + 1 \in \mathbb{F}_2[t]$ is irreducible.
  - (c) Show that $\mathbb{F}_2[t]/(t^5 + t^2 + 1)$ is a field and determine the inverse of $t^3 \bmod (t^5 + t^2 + 1)$ in this field.
  - (d) Determine the minimal polynomial of $t^3 \bmod (t^5 + t^2 + 1)$ over $\mathbb{F}_2$.